THE LITTLEWOOD-RICHARDSON RULE AND GELFAND-TSETLIN PATTERNS

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ABSTRACT. Using Gelfand-Tsetlin patterns as the main machinery of our analysis, we study the interrelationship of various combinatorial descriptions of the Littlewood-Richardson rule.

1. INTRODUCTION

1.1. Let us consider Schur polynomials s_μ, s_ν and s_λ in n variables labeled by partitions μ, ν and λ, respectively. *The Littlewood-Richardson (LR) coefficient* is the multiplicity $c^\lambda_{\mu,\nu}$ of s_ν in the product of s_μ and s_ν:

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda$$

and its description is called *the LR rule*.

The same number appears in the tensor product decomposition problem in the representation theory of the complex general linear group GL_n and Schubert calculus in the cohomology of the Grassmannians, and is also related to the eigenvalues of the sum of Hermitian matrices. For more details, we refer readers to [Fu00, HL12, Ta04, vL01].

1.2. The LR rule is usually stated in terms of combinatorial objects called *LR tableaux*. Recall that a Young tableau is a filling of the boxes of a Young diagram with positive integers. We shall use the English convention of drawing Young diagrams and tableaux as in [Fu97, St99].

Definition 1.1. *A tableau* T *on a skew Young diagram is called a LR tableau if it satisfies the following conditions:*

(1) *it is semistandard, that is, the entries in each row of* T *weakly increase from left to right, and the entries in each column strictly increase from top to bottom; and*
(2) *its reverse reading word is a Yamanouchi word (or lattice permutation). That is, in the word* $x_1x_2x_3 \ldots x_r$ *obtained by reading all the entries of* T *from left to right in each row starting from the bottom one, the sequence* $x_rx_{r-1}x_{r-2}\ldots x_s$ *contains at least as many* a*'s as it does* $(a+1)$*'s for all* $a \geq 1$.

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For example, the following is a LR tableau on a skew Young diagram $(11, 7, 5, 3)/(5, 3, 1)$

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
 & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
 & & & 1 & 2 & 2 & 2 \\
\cline{1-7}
 & 2 & 3 & 3 & 3 \\
\cline{1-5}
2 & 4 & 4 \\
\cline{1-3}
\end{array}$$

and its reverse reading word is 24423331222111111.

Remark 1.2. *(1) In this paper we assume each tableau's entries weakly increase from left to right in every row. (2) From the second condition in the above definition, which we will call the Yamanouchi condition, the* b*th row of a LR tableau does not contain any entries strictly bigger than* b *for all* $b \geq 1$.

The number of LR tableaux on the skew shape λ/μ with content ν is equal to the LR number $c^{\lambda}_{\mu,\nu}$. Here, the content $\nu = (\nu_1, \ldots, \nu_n)$ of a tableau means that the entry k appears ν_k times in the tableau for $k \geq 1$. See, for example, [Ma95, §I.9] and [HL12].

1.3. In this paper, we study variations of the semistandard and Yamanouchi conditions with an emphasis on dualities in combinatorial descriptions of the LR rule.

(1) In Theorem 2.5 and Theorem 3.2, we analyze *hives*, introduced by Knutson and Tao along with their honeycomb model [KT99, Bu00], in terms of Gelfand-Tsetlin(GT) patterns [GT50]. We then show how the interlacing conditions in GT patterns are intertwined to form the defining conditions of hives.

(2) In Theorem 4.4, we show that the semistandard and Yamanouchi conditions in LR tableaux are equivalent to, respectively, the interlacing and exponent conditions in *GZ schemes* introduced by Gelfand and Zelevinsky [GZ85]. These equivalences are obtained by matching t-arrays obtained from hives with t-arrays extracted from larger, truncated GT patterns.

(3) In Theorem 5.3, we show that the semistandard and Yamanouchi conditions in LR tableaux are equivalent to, respectively, the exponent and semistandard conditions in their *companion tableaux* introduced by van Leeuwen [vL01]. Here the correspondence between conditions is obtained by taking the transpose of matrices.

As a consequence, we obtain bijections between the families of combinatorial objects counting the LR number.

1.4. In [HTW05, HJLTW09], Howe and his collaborators constructed a polynomial model for the tensor product of representations in terms of two copies of the multi-homogeneous coordinate ring of the flag variety, and then studied its toric degeneration with the SAGBI-Gröbner method. Through the characterization of the leading monomials of highest weight vectors, their toric variety is encoded by the *LR cone* [PV05]. On the other hand, via toric degenerations, the flag variety may be described in terms of the lattice cone of GT patterns [GL96, Ki08, KM05]. These results led us to study the LR rule in terms of two sets of interlacing or semistandard conditions and to investigate the interrelationship of various combinatorial descriptions of the LR rule with GT patterns.

2. Hives and GT Patterns I

In this section, we define GT patterns, hives, and objects related to them. We also describe hives in terms of two copies of GT patterns.

2.1. We set, once and for all, three polynomial dominant weights of the complex general linear group GL_n, that is, the sequences of nonnegative integers:

$$\lambda = (\lambda_1, \dots, \lambda_n), \quad \mu = (\mu_1, \dots, \mu_n), \quad \nu = (\nu_1, \dots, \nu_n)$$

such that $\lambda_i \geq \lambda_{i+1}$, $\mu_i \geq \mu_{i+1}$, and $\nu_i \geq \nu_{i+1}$ for all i. We define the dual λ^* of λ to be

$$\lambda^* = (-\lambda_n, -\lambda_{n-1}, \dots, -\lambda_1),$$

and define μ^* and ν^* similarly.

2.2. Let us consider an array of integers, which we will call a t-*array*

$$T = \left(t_1^{(1)}, \dots, t_j^{(i)}, \dots, t_n^{(n)}\right) \in \mathbb{Z}^{n(n+1)/2}$$

where $1 \leq j \leq i \leq n$. We are particularly interested in the case when the entries of T are either all non-negative or all non-positive integers.

Definition 2.1. *A t-array $T = (t_j^{(i)}) \in \mathbb{Z}^{n(n+1)/2}$ is called a GT pattern for GL_n if it satisfies the interlacing conditions:*

$$\text{IC(1):} \qquad t_j^{(i+1)} \geq t_j^{(i)}$$
$$\text{IC(2):} \qquad t_j^{(i)} \geq t_{j+1}^{(i+1)}$$

for all i and j.

We shall draw a t-array in the reversed pyramid form. For example, a generic GT pattern for GL_5 is

$$\begin{array}{ccccccccc}
t_1^{(5)} & & t_2^{(5)} & & t_3^{(5)} & & t_4^{(5)} & & t_5^{(5)} \\
& t_1^{(4)} & & t_2^{(4)} & & t_3^{(4)} & & t_4^{(4)} & \\
& & t_1^{(3)} & & t_2^{(3)} & & t_3^{(3)} & & \\
& & & t_1^{(2)} & & t_2^{(2)} & & & \\
& & & & t_1^{(1)} & & & &
\end{array}$$

where the entries are weakly decreasing along the diagonals from left to right.

Then, the *dual array* $T^* = (s_j^{(i)})$ of T is the t-array obtained by reflecting T over a vertical line and then multiplying -1, i.e.,

$$s_j^{(i)} = -t_{i+1-j}^{(i)}$$

for all $1 \leq j \leq i \leq n$.

Definition 2.2. *For a t-array $T = (t_j^{(i)}) \in \mathbb{Z}^{n(n+1)/2}$,*

(1) *the kth row of T is $t^{(k)} = (t_1^{(k)}, t_2^{(k)}, \dots, t_k^{(k)}) \in \mathbb{Z}^k$ for $1 \leq k \leq n$. The type of T is its nth row;*

(2) *the weight of* T *is* $(w_1, w_2, \ldots, w_n) \in \mathbb{Z}^n$ *where* $w_1 = t_1^{(1)}$ *and*

$$w_i = \sum_{k=1}^{i} t_k^{(i)} - \sum_{k=1}^{i-1} t_k^{(i-1)} \quad \textit{for} \ \ 2 \leq i \leq n.$$

Note that if T is of type λ and weight $w \in \mathbb{Z}^n$, then T^* is of type λ^* and weight $-w$.

GT patterns were introduced by Gelfand and Tsetlin in [GT50] to label the weight basis elements of an irreducible representation of the general linear group. The weight of T is exactly the weight of the basis element labeled by T in the irreducible representation V_n^{μ} whose highest weight is $\mu = t^{(n)}$. It follows that the dual array T^* of T corresponds to a weight vector in the contragradiant representation of V_n^{μ}.

2.3. Let us consider an array of nonnegative integers, which we will call a h-*array*,

$$(h_{0,0}, \ldots, h_{a,b}, \ldots, h_{n,n}) \in \mathbb{Z}^{(n+1)(n+2)/2}$$

where $0 \leq a \leq b \leq n$ and $h_{0,0} = 0$.

Definition 2.3. *A hive for* GL_n *is a* h*-array* $H = (h_{a,b}) \in \mathbb{Z}^{(n+1)(n+2)/2}$ *satisfying the rhombus conditions:*

$$\begin{array}{lll} \textit{RC(1):} & (h_{a,b} + h_{a-1,b-1}) \geq (h_{a-1,b} + h_{a,b-1}) & \textit{for} \ \ 1 \leq a < b \leq n, \\ \textit{RC(2):} & (h_{a-1,b} + h_{a,b}) \geq (h_{a,b+1} + h_{a-1,b-1}) & \textit{for} \ \ 1 \leq a \leq b < n, \\ \textit{RC(3):} & (h_{a,b} + h_{a,b+1}) \geq (h_{a+1,b+1} + h_{a-1,b}) & \textit{for} \ \ 1 \leq a \leq b < n. \end{array}$$

We shall draw a h-array in the pyramid form. For example, a generic hive for GL_3 is shown below.

$$\begin{array}{ccccccc} & & & h_{0,0} & & & \\ & & h_{0,1} & & h_{1,1} & & \\ & h_{0,2} & & h_{1,2} & & h_{2,2} & \\ h_{0,3} & & h_{1,3} & & h_{2,3} & & h_{3,3} \end{array}$$

The rhombus conditions RC(1), RC(2), and RC(3) then say that, for each fundamental rhombus of one of the following forms,

$$\begin{array}{ccccccccccc} & & & & & A & & & & & \\ & O' & & A' & & O & & O' & & A' & & O \\ A & & O & & , & & A' & & , & & O' & & A \end{array}$$

the sum of entries at the obtuse corners is bigger than or equal to the sum of entries at the acute corners, i.e., $O + O' \geq A + A'$.

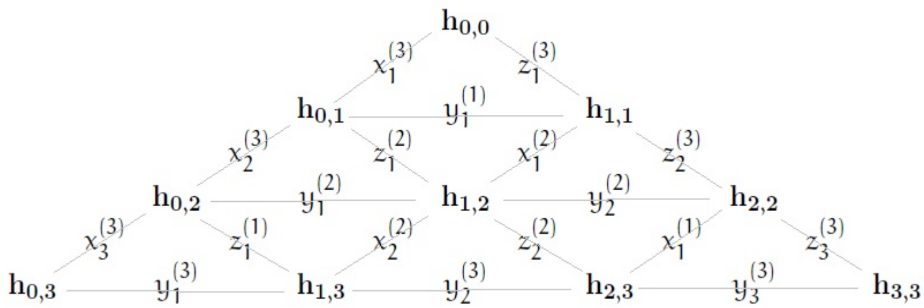

FIGURE 1. A h-array and its three derived t-arrays.

For polynomial dominant weights μ, ν, and λ of GL_n, we let $\mathcal{H}(\mu,\nu,\lambda)$ denote the set of all h-arrays such that

$$
\begin{aligned}
\mu &= (h_{0,1}-h_{0,0}, h_{0,2}-h_{0,1}, \dots, h_{0,n}-h_{0,n-1}),\\
\nu &= (h_{1,n}-h_{0,n}, h_{2,n}-h_{1,n}, \dots, h_{n,n}-h_{n-1,n}),\\
\lambda &= (h_{1,1}-h_{0,0}, h_{2,2}-h_{1,1}, \dots, h_{n,n}-h_{n-1,n-1}).
\end{aligned}
\tag{2.3.1}
$$

That is, the three boundary sides of $H \in \mathcal{H}(\mu,\nu,\lambda)$ are fixed:

$$
\begin{aligned}
h_{0,i} &= \mu_1+\mu_2+\cdots+\mu_i\\
h_{i,n} &= \sum_{j=1}^{n}\mu_j+\nu_1+\nu_2+\cdots+\nu_i\\
h_{i,i} &= \lambda_1+\lambda_2+\cdots+\lambda_i
\end{aligned}
$$

for $1 \le i \le n$. Recall that we always set $h_{0,0}=0$. Let $\mathcal{H}^\circ(\mu,\nu,\lambda)$ be the subset of $\mathcal{H}(\mu,\nu,\lambda)$ satisfying the rhombus conditions. This is the set of hives whose boundaries are described by (2.3.1).

Hives were introduced by Knutson and Tao in [KT99] along with their honeycomb model to prove the saturation conjecture. In particular, the number of hives in $\mathcal{H}^\circ(\mu,\nu,\lambda)$ is equal to the LR number $c^{\lambda}_{\mu,\nu}$. See also [Bu00, KTW04, PV05].

2.4. For each h-array $H=(h_{a,b}) \in \mathbb{Z}^{(n+1)(n+2)/2}$, let us define its *derived* t-*arrays*

$$T_1=(x_j^{(i)}), \quad T_2=(y_j^{(i)}), \quad T_3=(z_j^{(i)})$$

whose entries are obtained from the differences of adjacent entries of H. More specifically, for each fundamental triangle in H,

$$
\begin{array}{ccc}
 & h_{a,b} & \\
 & & \\
h_{a,b+1} & & h_{a+1,b+1}
\end{array}
$$

the entries of the derived t-arrays $(x_j^{(i)})$, $(y_j^{(i)})$, and $(z_j^{(i)})$ are

$$\begin{aligned} x_{b+1-a}^{(n-a)} &= h_{a,b+1} - h_{a,b} \quad \text{(SW–NE direction)} \\ y_{a+1}^{(b+1)} &= h_{a+1,b+1} - h_{a,b+1} \quad \text{(E–W direction)} \\ z_{a+1}^{(n+a-b)} &= h_{a+1,b+1} - h_{a,b} \quad \text{(SE–NW direction)} \end{aligned} \tag{2.4.1}$$

for $0 \le a \le b \le n-1$.

This rather involved indexing is to make the entries of the derived arrays compatible with those of GT patterns. We may visualize the derived t-arrays by placing their entries between the entries of the h-array used to compute them. For example, if $n = 3$, then a h-array and its three derived t-arrays may be drawn as Figure 1.

2.5. The rhombus conditions for h-arrays are closely related to the interlacing conditions for their derived t-arrays.

Proposition 2.4. *Let $T_k = T_k(H)$ be a derived t-array of a h-array H for $k = 1, 2, 3$.*

1. *H satisfies RC(1) if and only if T_1 satisfies IC(2) and T_2 satisfies IC(1).*
2. *H satisfies RC(2) if and only if T_1 and T_3 satisfy IC(1).*
3. *H satisfies RC(3) if and only if T_2 and T_3 satisfy IC(2).*
4. *T_3 satisfies IC(1) if and only if T_1 satisfies IC(1).*
5. *T_3 satisfies IC(2) if and only if T_2 satisfies IC(2).*

Proof. Let us consider five adjacent entries of H of the forms

$$\begin{array}{ccc} & Z_1 & \\ Y_1 & & W_1 \\ X_1 & & V_1 \end{array}, \quad \begin{array}{ccc} & Y_2 & & W_2 & \\ X_2 & & V_2 & & U_2 \end{array}, \quad \begin{array}{ccc} & Z_3 & \\ Y_3 & & W_3 \\ & V_3 & & U_3 \end{array}.$$

Then, in the first and the third ones, RC(2) says that $Y_i + W_i \ge Z_i + V_i$ for $i = 1$ and 3. This is equivalent to $Y_1 - Z_1 \ge V_1 - W_1$ and $W_3 - Z_3 \ge V_3 - Y_3$, which are IC(1) for T_1 and T_3, respectively. This proves the statement (2). The statements (1) and (3) can be shown similarly.

Next, let us consider fundamental rhombi of the following forms in H

$$\begin{array}{ccc} & K & \\ L & & N \\ & M & \end{array}, \quad \begin{array}{ccc} P & & S \\ & Q & & R. \end{array}$$

Note that $N - K \ge M - L$ if and only if $L - K \ge M - N$, which proves (4). Similarly, $P - Q \ge S - R$ if and only if $P - S \ge Q - R$, which proves (5). $\square$

Suppose a h-array H satisfies RC(1), RC(2), and RC(3). Then, by the statements (1) and (2) of Proposition 2.4, $T_1(H)$ satisfies IC(1) and IC(2). Similarly, by the statements (1) and (3), $T_2(H)$ satisfies IC(1) and IC(2). This shows that $T_1(H)$ and $T_2(H)$ are GT

patterns. Conversely, if $T_1(H)$ and $T_2(H)$ are GT patterns, then, by the statements (4) and (5), $T_3(H)$ is also a GT pattern. This means all three derived t-arrays satisfy both IC(1) and IC(2), and therefore, from the statements (1), (2), and (3), H is a hive.

Theorem 2.5. *For a* h*-array* $H \in \mathbb{Z}^{(n+1)(n+2)/2}$ *and its derived* t*-arrays* $T_1(H)$ *and* $T_2(H)$*,* H *is a hive if and only if* $T_1(H)$ *and* $T_2(H)$ *are GT patterns for* GL_n*.*

Note that in the above result $T_1(H)$ and $T_2(H)$ are not independent. Let $T_1 = (x_j^{(i)})$ and $T_2 = (y_j^{(i)})$ be the derived t-arrays of a h-array H. Then, for each rhombus of the form

$$\begin{matrix} & & B & & A \\ C & & D & & \end{matrix}$$

we have $(D - C) + (C - B) = (D - A) + (A - B)$, or

$$(C - B) - (D - A) = (A - B) - (D - C)$$

which is, using (2.4.1),

$$x_{b-a}^{(n-a-1)} - x_{b+1-a}^{(n-a)} = y_{a+1}^{(b+1)} - y_{a+1}^{(b)} \tag{2.5.1}$$

for $0 \le a < b < n$.

2.6. We remark that hives (respectively, GT patterns) for GL_n form a subsemigroup of $\mathbb{Z}^{(n+1)(n+2)/2}$ (respectively, $\mathbb{Z}^{n(n+1)/2}$). Then, Theorem 2.5 and (2.5.1) show that the semigroup of hives is a fiber product of, over $\mathbb{Z}_{\ge 0}^{n(n-1)/2}$, two affine semigroups S_{GT}^1 and S_{GT}^2 of GT patterns with respect to

$$\phi_k : S_{GT}^k \longrightarrow \mathbb{Z}_{\ge 0}^{n(n-1)/2}$$

such that, for $0 \le a < b < n$,

$$\begin{aligned} \phi_1(T_1) &= \left(\ldots, x_{b-a}^{(n-a-1)} - x_{b+1-a}^{(n-a)}, \ldots\right) \\ \phi_2(T_2) &= \left(\ldots, y_{a+1}^{(b+1)} - y_{a+1}^{(b)}, \ldots\right) \end{aligned}$$

where $T_1 = (x_j^{(i)}) \in S_{GT}^1$ and $T_2 = (y_j^{(i)}) \in S_{GT}^2$.

3. Hives and GT Patterns II

In this section, we study the set $\mathcal{H}^\circ(\mu, \nu, \lambda)$ of hives with a given boundary condition in terms of a single GT pattern.

3.1. Gelfand and Zelevinsky counted the LR number $c_{\mu,\nu}^{\lambda}$ with GT patterns of type μ and weight $\lambda - \nu$ satisfying the following additional condition.

Lemma 3.1. [GZ85] *For a* t*-array* $T = (t_j^{(i)}) \in \mathbb{Z}^{n(n+1)/2}$*, we define its exponents as*

$$\varepsilon_j^{(i)}(T) = \sum_{1 \le h < j} (t_h^{(i+1)} - 2t_h^{(i)} + t_h^{(i-1)}) + (t_j^{(i+1)} - t_j^{(i)}).$$

Then the cardinality of the set $\mathsf{GZ}(\mu, \lambda - \nu, \nu)$ *of all GT patterns* T *of type* μ *with weight* $\lambda - \nu$ *such that, for all* i *and* j,

$$\varepsilon_j^{(i)}(T) \leq \nu_i - \nu_{i+1}$$

is equal to the LR number $c_{\mu,\nu}^{\lambda}$.

The elements of $\mathsf{GZ}(\mu, \lambda - \nu, \nu)$ will be called *GZ schemes.*

3.2. Note that, for a h-array H, since the derived t-arrays are defined from the differences of the entries in H, if the boundaries of H are fixed, then any one of the derived t-array of H uniquely determines H. Moreover, we can characterize the derived t-arrays as follows.

Theorem 3.2. *For a* h*-array* H *in* $\mathcal{H}(\mu, \nu, \lambda)$, *consider its derived* t*-arrays* $T_1(H)$ *and* $T_2(H)$.

(1) H *is a hive if and only if* $T_1^*(H) = (T_1(H))^*$ *is a GZ scheme in* $\mathsf{GZ}(\mu^*, \lambda^* - \nu^*, \nu^*)$;
(2) H *is a hive if and only if* $T_2(H)$ *is a GZ scheme in* $\mathsf{GZ}(\nu, \lambda - \mu, \mu)$.

Note that this theorem, in particular, gives bijections between hives and GZ schemes:

$$\begin{array}{ccc} \mathcal{H}^{\circ}(\mu, \nu, \lambda) & \longrightarrow & \mathsf{GZ}(\mu^*, \lambda^* - \nu^*, \nu^*) \\ H & \longmapsto & T_1^*(H) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{H}^{\circ}(\mu, \nu, \lambda) & \longrightarrow & \mathsf{GZ}(\nu, \lambda - \mu, \mu) \\ H & \longmapsto & T_2(H) \end{array} .$$

For the rest of this section, we will prove Theorem 3.2 by showing the following.

(a) $T_1^*(H)$ satisfies IC(2) if and only if $\varepsilon_j^{(i)}(T_2(H)) \leq \mu_i - \mu_{i+1}$;
(b) $T_1^*(H)$ satisfies IC(1) if and only if $T_2(H)$ satisfies IC(1);
(c) $T_1^*(H)$ satisfies $\varepsilon_j^{(i)}(T_1^*(H)) \leq \nu_i^* - \nu_{i+1}^*$ if and only if $T_2(H)$ satisfies IC(2).

The weights of the derived t arrays will also be computed.

3.3. Let us first compute the weights of $T_1(H)$ and $T_2(H)$ for $H \in \mathcal{H}(\mu, \nu, \lambda)$.

Lemma 3.3. *For a* h*-array* $H = (h_{a,b}) \in \mathcal{H}(\mu, \nu, \lambda)$,

(1) *the weight of* $T_1(H)$ *is* $\nu^* - \lambda^*$, *i.e.*,

$$(\lambda_n - \nu_n, \lambda_{n-1} - \nu_{n-1}, \ldots, \lambda_1 - \nu_1)$$

therefore, the weight of $T_1^*(H)$ *is* $\lambda^* - \nu^*$;

(2) *the weight of* $T_2(H)$ *is* $\lambda - \mu$, *i.e.*,

$$(\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots, \lambda_n - \mu_n).$$

Proof. We will prove the second statement. The proof of the first case is similar. From Definition 2.2, (2.4.1) and the expressions for λ and μ in terms of the h-array elements it follows

$$w_1 = y_1^{(1)} = h_{1,1} - h_{0,1} = (h_{1,1} - h_{0,0}) + (h_{0,0} - h_{0,1}) = \lambda_1 - \mu_1.$$

Using the same approach for w_i, $i \geq 2$, we see

$$\begin{aligned} w_i &= \sum_{k=1}^{i} y_k^{(i)} - \sum_{k=1}^{i-1} y_k^{(i-1)} \\ &= \sum_{k=1}^{i} (h_{k,i} - h_{k-1,i}) - \sum_{k=1}^{i-1} (h_{k,i-1} - h_{k-1,i-1}) \\ &= (h_{i,i} - h_{0,i}) - (h_{i-1,i-1} - h_{0,i-1}) \\ &= \lambda_i - \mu_i. \end{aligned}$$

Therefore $w_i = \lambda_i - \mu_i$ for all i, and the weight of $T_2(H)$ is $\lambda - \mu$. □

3.4. Next, we study the relations between the interlacing conditions and the exponents conditions for derived arrays. Note that, from the definition of dual arrays, a t-array T satisfies IC(1) if and only if T^* satisfies IC(2), and T satisfies IC(2) if and only if T^* satisfies IC(1).

Proposition 3.4. *For a* h*-array* $H = (h_{a,b}) \in \mathcal{H}(\mu, \nu, \lambda)$ *and its derived* t*-arrays* $T_1(H) = (x_j^{(i)})$ *and* $T_2(H) = (y_j^{(i)})$*,* $T_1(H)$ *satisfies IC(1) if and only if* $\varepsilon_j^{(i)}(T_2(H)) \leq \mu_i - \mu_{i+1}$.

Proof. Let us assume $j > 1$. Then the exponent of $T_2(H)$,

$$\varepsilon_j^{(i)}(T_2(H)) = \sum_{1 \leq h < j} \left((y_h^{(i+1)} - y_h^{(i)}) - (y_h^{(i)} - y_h^{(i-1)}) \right) + \left(y_j^{(i+1)} - y_j^{(i)} \right)$$

can be rewritten in terms of the entries of $T_1(H)$. By using (2.5.1),

$$\begin{aligned} \varepsilon_j^{(i)}(T_2(H)) \quad = \quad & \sum_{1 \leq h < j} \left((x_{i-h+1}^{(n-h)} - x_{i-h+2}^{(n-h+1)}) - (x_{i-h}^{(n-h)} - x_{i-h+1}^{(n-h+1)}) \right) \\ & + \left(x_{i-j+1}^{(n-j)} - x_{i-j+2}^{(n-j+1)} + y_j^{(i)} \right) - \left(x_{i-j}^{(n-j)} - x_{i-j+1}^{(n-j+1)} + y_j^{(i-1)} \right) \end{aligned}$$

and we see that parts of the consecutive terms cancel to give

$$\varepsilon_j^{(i)}(T_2(H)) = \left(x_i^{(n)} - x_{i+1}^{(n)} \right) + \left(x_{i-j+1}^{(n-j)} - x_{i-j}^{(n-j)} + y_j^{(i)} - y_j^{(i-1)} \right). \tag{3.4.1}$$

Now note that the interlacing condition IC(1) for $T_1(H)$ implies $x_{i-j+1}^{(n-j+1)} \geq x_{i-j+1}^{(n-j)}$ or equivalently, by using (2.5.1),

$$x_{i-j}^{(n-j)} \quad \geq \quad \left(x_{i-j+1}^{(n-j)} + y_j^{(i)} - y_j^{(i-1)} \right)$$

therefore

$$0 \quad \geq \quad \left(x_{i-j+1}^{(n-j)} - x_{i-j}^{(n-j)} + y_j^{(i)} - y_j^{(i-1)} \right).$$

Hence, from (3.4.1), the interlacing condition IC(1) for $T_1(H)$ is equivalent to

$$\varepsilon_j^{(i)}(T_2(H)) \leq \left(x_i^{(n)} - x_{i+1}^{(n)} \right) = \mu_i - \mu_{i+1}.$$

The case $j = 1$ can be shown similarly for all i. □

Proposition 3.5. *For a* h-*array* $H = (h_{a,b}) \in \mathcal{H}(\mu, \nu, \lambda)$ *and its derived* t-*arrays* $T_1(H) = (x_j^{(i)})$ *and* $T_2(H) = (y_j^{(i)})$, $T_1(H)$ *satisfies IC(2) if and only if* $T_2(H)$ *satisfies IC(1).*

Proof. Using the equality (2.5.1),

$$\left(x_j^{(i)} \geq x_{j+1}^{(i+1)}\right) \quad \text{if and only if} \quad \left(y_{n-i}^{(n-i+j)} \geq y_{n-i}^{(n-i+j-1)}\right)$$

and therefore, by setting $i' = n - i + j - 1$ and $j' = n - i$, we have

$$\left(x_j^{(i)} \geq x_{j+1}^{(i+1)}\right) \quad \text{if and only if} \quad \left(y_{j'}^{(i'+1)} \geq y_{j'}^{(i')}\right)$$

for $1 \leq j \leq i \leq n-1$ and $1 \leq j' \leq i' \leq n-1$. This shows that IC(2) holds for $T_1(H)$ if and only if IC(1) holds for $T_2(H)$. □

Proposition 3.6. *For a* h-*array* $H = (h_{a,b}) \in \mathcal{H}(\mu, \nu, \lambda)$ *and its derived* t-*arrays* $T_1(H) = (x_j^{(i)})$ *and* $T_2(H) = (y_j^{(i)})$, $T_1^*(H)$ *satisfies* $\varepsilon_j^{(i)}(T_1^*(H)) \leq \nu_i^* - \nu_{i+1}^*$ *if and only if* $T_2(H)$ *satisfies IC(2).*

Proof. Let us assume $j > 1$. Write the exponents of $T_1^*(H) = (s_j^{(i)})$ using $s_j^{(i)} = -x_{i+1-j}^{(i)}$.

$$\begin{aligned}
\varepsilon_j^{(i)}(T_1^*(H)) &= \sum_{1 \leq h < j} \left(-x_{i-h+2}^{(i+1)} + 2x_{i-h+1}^{(i)} - x_{i-h}^{(i-1)}\right) + \left(-x_{i-j+2}^{(i+1)} + x_{i-j+1}^{(i)}\right) \\
&= \sum_{1 \leq h < j} \left((x_{i-h+1}^{(i)} - x_{i-h+2}^{(i+1)}) - (x_{i-h}^{(i-1)} - x_{i-h+1}^{(i)})\right) + \left(x_{i-j+1}^{(i)} - x_{i-j+2}^{(i+1)}\right)
\end{aligned}$$

Then, using the identity (2.5.1), we can rewrite the exponents in terms of the entries of $T_2(H)$ as

$$\begin{aligned}
\varepsilon_j^{(i)}(T_1^*(H)) &= \sum_{1 \leq h < j} \left((y_{n-i}^{(n-h+1)} - y_{n-i}^{(n-h)}) - (y_{n-i+1}^{(n-h+1)} - y_{n-i+1}^{(n-h)})\right) + \left(y_{n-i}^{(n-j+1)} - y_{n-i}^{(n-j)}\right) \\
&\leq \sum_{1 \leq h < j} \left((y_{n-i}^{(n-h+1)} - y_{n-i}^{(n-h)}) - (y_{n-i+1}^{(n-h+1)} - y_{n-i+1}^{(n-h)})\right) + \left(y_{n-i}^{(n-j+1)} - y_{n-i+1}^{(n-j+1)}\right)
\end{aligned}$$

where the inequality is by IC(2): $y_{n-i}^{(n-j)} \geq y_{n-i+1}^{(n-j+1)}$ in $T_2(H)$. Parts of the consecutive terms in the right hand side cancel to give

$$\begin{aligned}
\varepsilon_j^{(i)}(T_1^*(H)) &\leq \left((y_{n-i}^{(n)} - y_{n-i}^{(n-j+1)}) - (y_{n-i+1}^{(n)} - y_{n-i+1}^{(n-j+1)})\right) + \left(y_{n-i}^{(n-j+1)} - y_{n-i+1}^{(n-j+1)}\right) \\
&= \left(y_{n-i}^{(n)} - y_{n-i+1}^{(n)}\right) = \nu_{n-i} - \nu_{n-i+1} = \nu_i^* - \nu_{i+1}^*.
\end{aligned}$$

So the interlacing condition IC(2) for $T_2(H)$ is equivalent to $\varepsilon_j^{(i)}(T_1^*(H)) \leq \nu_i^* - \nu_{i+1}^*$ as required. The case $j = 1$ can be shown similarly for all i. □

3.5. Suppose we have a hive H. From Lemma 3.3, the weights of $T_1^*(H)$ and $T_2(H)$ are $\lambda^* - \nu^*$ and $\lambda - \mu$, respectively. Theorem 2.5 states that H is a hive if and only if $T_1(H)$ and $T_2(H)$, and hence $T_1^*(H)$ and $T_2(H)$, satisfy both IC(1) and IC(2). Therefore since H is a hive, Proposition 3.4 and Proposition 3.6 imply $T_1^*(H)$ and $T_2(H)$ satisfy the exponent conditions, and consequently they are GZ schemes in $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$ and $GZ(\nu, \lambda - \mu, \mu)$, respectively.

Conversely, if $T_1^*(H)$ is a GZ scheme from $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$ it satisfies IC(1), IC(2), and the exponent condition, thus from Propositions 3.4 – 3.6, $T_2(H)$ is a GZ scheme. In particular, $T_1(H)$ and $T_2(H)$ are GT patterns, meaning H is a hive by Theorem 2.5. Similarly, if $T_2(H) \in GZ(\nu, \lambda - \mu, \mu)$, then H is a hive. This proves Theorem 3.2.

4. LR Tableaux and GT Patterns I

In this section, we show that the semistandard and Yamanouchi conditions for tableaux are equivalent to, respectively, the interlacing and exponent conditions for t-arrays. This correspondence is obtained by extracting t-arrays from larger, truncated GT patterns. As a result, we obtain a bijection between LR tableaux and GZ schemes.

4.1. A non-skew semistandard tableau Y can be uniquely determined by its associated matrix $(a_{i,j}(Y))$ where

$$a_{i,j}(Y) = \text{ the number of } i\text{'s in the } j\text{th row} \tag{4.1.1}$$

for all $1 \le i, j \le n$. Note that $a_{i,j}(Y) = 0$ for $i < j$. We also note that $\sum_{k=1}^{n} a_{k,j}(Y)$ for $1 \le j \le n$ give the shape of the tableau Y, and $\sum_{k=1}^{n} a_{i,k}(Y)$ for $1 \le i \le n$ give the content of Y. We remark that if Y is a semistandard tableau on the skew shape λ/μ, then the $a_{i,j}(Y)$'s are well defined, and the $a_{i,j}(Y)$'s with λ or μ uniquely define Y. It is possible to develop the theory of tableaux exclusively in terms of their associated matrices. See [DK05] for this direction.

For a GT pattern $T = (t_j^{(i)})$ of type λ with non-negative entries, let us consider a Young tableau Y_T of shape λ such that

$$a_{i,j}(Y_T) = t_j^{(i)} - t_j^{(i-1)} \tag{4.1.2}$$

for $1 \le i, j \le n$ with the conventions

$$t_j^{(i)} = 0 \quad \text{for} \quad j > i \ge 0.$$

This correspondence provides a bijection between the set of GT patterns of type λ and the set of semistandard Young tableaux of shape λ whose entries are from $\{1, 2, \ldots, n\}$. The GT pattern $T_Y = (t_j^{(i)})$ corresponding to a semistandard tableau Y in this bijection is then given by

$$t_j^{(i)} = \sum_{k=1}^{i} a_{k,j}(Y) \tag{4.1.3}$$

for $1 \le j \le i \le n$. Since $a_{k,j}(Y) = 0$ for $k < j$ in every non-skew semistandard tableau Y, we can in fact write this as

$$t_j^{(i)} = \sum_{k=j}^{i} a_{k,j}(Y). \tag{4.1.4}$$

See, e.g., [GW09, §8.1.2] or [Ki08].

4.2. Under this bijection, the content of Y_T is equal to the weight of T. We also note that under this bijection, the semistandard condition in Y_T is implied by the interlacing condition in T and vice versa (cf. Remark 4.2). Moreover, this bijection can be extended to their skew versions.

Lemma 4.1. *There is a bijection between the set of skew semistandard Young tableaux of shape λ/μ with entries from $\{1,2,\dots,n\}$ and the set of GT patterns for GL_{2n} whose type is $\lambda' = (\lambda_1,\dots,\lambda_n,0,\cdots,0) \in \mathbb{Z}^{2n}$ and whose kth row is $(\mu_1,\mu_2,\dots,\mu_k)$ for $1 \le k \le n$.*

Proof. For a given semistandard Young tableau Y of shape λ/μ, replace the i entries with $(n+i)$'s for $1 \le i \le n$, then fill in the empty boxes in the ℓth row of Y with ℓ's for $1 \le \ell \le n$. Then this process uniquely determines a non-skew semistandard Young tableau of shape λ with entries from $\{1,2,\dots,2n\}$, and under the bijection given by (4.1.2), its corresponding GT pattern for GL_{2n} is the one described in the statement. □

4.3. Let us express the semistandard condition for a tableau Y in terms of the $a_{i,j}(Y)$ defined in (4.1.1). By rearranging the entries in each row if necessary, we can always make the entries of Y weakly increasing along each row from left to right. The strictly increasing condition on the columns of Y can then be rephrased as follows: the number of entries up to ℓ in the $(m+1)$th row is not bigger than the number of entries up to $(\ell-1)$ in the mth row, i.e.,

$$\sum_{k=1}^{\ell-1} a_{k,m}(Y) \ge \sum_{k=1}^{\ell} a_{k,m+1}(Y) \tag{4.3.1}$$

for $1 \le \ell \le n$ and $1 \le m < n$. Here, if $\ell = 1$, then the left hand side is 0 as an empty sum and the inequality implies that $a_{1,m+1}(Y) = 0$ for $m \ge 1$. Inductively, we can obtain $a_{i,m+1}(Y) = 0$ for $m \ge i$ from the inequality with $\ell = i$. This shows that for a semistandard Young tableau Y, $a_{i,j}(Y) = 0$ for $j > i$, as we noted after (4.1.1).

Remark 4.2. *By using the conversion formula* (4.1.3)*, one can directly compute that IC(2) on a GT pattern T is equivalent to the semistandard condition* (4.3.1) *in Y_T corresponding to T. On the other hand, IC(1) in T is equivalent to a rather trivial condition $a_{i,j}(Y_T) \ge 0$ for all i,j.*

If Y is a skew tableau of shape λ/μ, then, using the same argument as for (4.3.1), it is straightforward to see that we can make Y semistandard by rearranging elements along each row if and only if

$$\mu_{m+1} + \sum_{k=1}^{\ell} a_{k,m+1}(Y) \le \mu_m + \sum_{k=1}^{\ell-1} a_{k,m}(Y) \tag{4.3.2}$$

for $1 \le \ell \le n$ and $1 \le m < n$. The Yamanouchi condition in a LR tableau Y can be expressed as

$$\sum_{k=1}^{j} a_{i+1,k}(Y) \le \sum_{k=1}^{j-1} a_{i,k}(Y) \tag{4.3.3}$$

for $1 \leq j \leq n$ and $1 \leq i < n$. Here, if $j = 1$, then the right hand side is 0 as an empty sum and the inequality implies that $a_{i+1,1}(Y) = 0$ for $i \geq 1$. Inductively, we can obtain $a_{i+1,\ell}(Y) = 0$ for $i \geq \ell$ from the inequality with $j = \ell$. This shows that for an LR tableau Y, $a_{i,j}(Y) = 0$ for $i > j$, as we noted in Remark 1.2 (2).

4.4. We remark that the GT pattern for GL_n whose kth row is $(\mu_1, \mu_2, \ldots, \mu_k)$ for $1 \leq k \leq n$ corresponds to the highest weight vector of the representation V_n^{μ} labeled by a Young diagram μ. In fact, the GT patterns described in Lemma 4.1 encode the weight vectors of $V_{2n}^{\lambda'}$, which are the highest weight vector for V_n^{μ} under the branching of GL_{2n} down to GL_n. Since the bottom $n-1$ rows of the GT patterns for GL_{2n} described in the above lemma are completely determined by the nth row, we can omit those $n-1$ rows and focus on the top $n+1$ rows. Such GT patterns will be called *truncated.*

Example 4.3. *Let* $\lambda = (11, 7, 5, 3)$, $\mu = (5, 3, 1, 0)$, *and* $\nu = (7, 5, 3, 2)$. *The LR tableau from* $LR(\lambda/\mu, \nu)$

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
 & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
 & & & 1 & 2 & 2 & 2 \\
\cline{1-7}
 & 2 & 3 & 3 & 3 \\
\cline{1-5}
2 & 4 & 4 \\
\cline{1-3}
\end{array}
$$

considered as an object for GL_4*, corresponds to the following truncated GT pattern.*

$$
\begin{array}{ccccccccccccccc}
11 & & 7 & & 5 & & 3 & & 0 & & 0 & & 0 & & 0 \\
 & 11 & & 7 & & 5 & & 1 & & 0 & & 0 & & 0 & \\
 & & 11 & & 7 & & 2 & & 1 & & 0 & & 0 & & \\
 & & & 11 & & 4 & & 1 & & 0 & & 0 & & & \\
 & & & & 5 & & 3 & & 1 & & 0 & & & &
\end{array}
$$

Note that in the above example, we can divide the truncated GT pattern into 3 GT patterns of type λ, μ, and $(0, \ldots, 0)$.

$$
\left[\begin{array}{ccccccc}
11 & & 7 & & 5 & & 3 \\
 & 11 & & 7 & & 5 & \\
 & & 11 & & 7 & & \\
 & & & 11 & & &
\end{array}\right],
\left[\begin{array}{ccccccc}
5 & & 3 & & 1 & & 0 \\
 & 4 & & 1 & & 0 & \\
 & & 2 & & 1 & & \\
 & & & 1 & & &
\end{array}\right],
\left[\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
 & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}\right]
$$

See also Figure 2 below.

4.5. We now study LR tableaux in terms of the interlacing conditions in their corresponding GT patterns.

Theorem 4.4. *There is a bijection* ϕ *between* $LR(\lambda/\mu, \nu)$ *and* $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$. *In particular, the semistandard and Yamanouchi conditions in* $L \in LR(\lambda/\mu, \nu)$ *are equivalent to, respectively, the semistandard and exponent conditions in* $\phi(L) \in GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$.

Proof. Let $L \in LR(\lambda/\mu, \nu)$ be given. Then, its corresponding GT pattern $T = (t_j^{(i)})$ for GL_{2n}, after removing the bottom $n-1$ rows, can be identified with a truncated GT pattern for GL_{2n} with $n+1$ rows. Furthermore, the truncated pattern for L can be

divided into three subtriangular arrays T_X, T_Y and T_Z, as in Figure 2. Note that these are the same size.

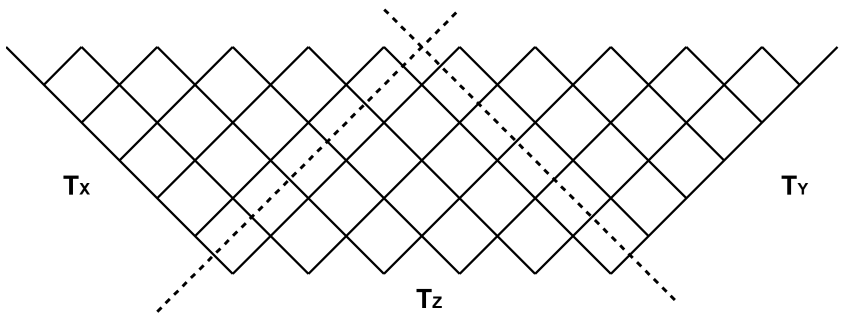

FIGURE 2. Dividing a truncated GT pattern into 3 subpatterns.

Then, the upper left subarray T_X is completely determined by λ because of the Yamanouchi condition (see Remark 1.2 (2)). The upper right subarray T_Y contains only zero. Therefore, with given λ, μ, and ν, the LR tableau $L \in LR(\lambda/\mu, \nu)$ is uniquely determined by T_Z. We want to show that the dual array T_Z^* of T_Z is an element of $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$, and from that establish a bijection

$$\begin{array}{rcl} LR(\lambda/\mu, \nu) & \longrightarrow & GZ(\mu^*, \lambda^* - \nu^*, \nu^*) \\ L & \longmapsto & T_Z^* \end{array} .$$

Let us rewrite the middle subarray T_Z as follows by reflecting it over a horizontal line.

$$T_Z = \begin{array}{ccccccccc} t_1^{(n)} & & t_2^{(n)} & & \cdots & & t_{n-1}^{(n)} & & t_n^{(n)} \\ & t_2^{(n+1)} & & t_3^{(n+1)} & & \cdots & & t_n^{(n+1)} & \\ & & t_3^{(n+2)} & & & & t_n^{(n+2)} & & \\ & & & \ddots & & \iddots & & & \\ & & & & t_n^{(2n-1)} & & & & \end{array}$$

Then $\mu_i = t_i^{(n)}$ for $1 \le i \le n$ and T_Z satisfies the interlacing conditions induced from the truncated GT pattern T, which are assured by the semistandardness of L. Therefore T_Z is a GT pattern of type μ. From the fact that the weights of T_X, T_Y, and T are $(\lambda_1, \ldots, \lambda_n)$, $(0, \ldots, 0)$, and $(\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n)$ respectively, it is easy to show that the weight of T_Z is $\nu^* - \lambda^*$. Hence its dual T_Z^* is a GT pattern (see §3.4) of type μ^* and weight $\lambda^* - \nu^*$. Next, we want to show that T_Z^* satisfies the exponent conditions.

Let $a_{i,j} = a_{i,j}(L)$, i.e., be the number of i's in the jth row of L for all i and j. Then

$$a_{i,j} = t_j^{(n+i)} - t_j^{(n+i-1)} \quad \text{and } a_{k,k} = \lambda_k - t_k^{(n+k-1)} \tag{4.5.1}$$

for $1 \le i < j \le n$ and $1 \le k \le n$. Since the content of L is ν with $\nu_q = \sum_{k=1}^n a_{q,k}$ for $1 \le q \le n$, we can write

$$(-\nu_{i+1}) - (-\nu_i) = \sum_{k=1}^n (a_{i,k} - a_{i+1,k}) \tag{4.5.2}$$

for $1 \le i < n$.

On the other hand, from the Yamanouchi condition (4.3.3) in L, we have

$$\sum_{k=1}^{j} a_{i+1,k} \le \sum_{k=1}^{j-1} a_{i,k} \quad \text{or equivalently,} \quad a_{i+1,j} \le \sum_{k=1}^{j-1} (a_{i,k} - a_{i+1,k}).$$

Then, using this inequality, (4.5.2) becomes

$$(-\nu_{i+1}) - (-\nu_i) \ge \sum_{k=j+1}^{n} (a_{i,k} - a_{i+1,k}) + a_{i,j}$$

and the right hand side is, via (4.5.1), the exponents of T_Z^*. Therefore, $T_Z^* \in GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$. □

Theorem 4.4 and Theorem 3.2 give a bijection between the set of LR tableaux and the set of hives.

$$\begin{array}{ccccc} & & GZ(\mu^*, \lambda^* - \nu^*, \nu^*) & & \\ & \swarrow\!\nearrow & & \searrow\!\nwarrow & \\ LR(\lambda/\mu, \nu) & & & & \mathcal{H}^\circ(\mu, \nu, \lambda) \end{array}$$

Corollary 4.5. *There is a bijection between* $LR(\lambda/\mu, \nu)$ *and* $\mathcal{H}^\circ(\mu, \nu, \lambda)$.

Proof. For $L \in LR(\lambda/\mu, \nu)$, we compute the corresponding truncated GT pattern and its middle subarray T_Z. Then, by Theorem 4.4, its dual T_Z^* belongs to $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$. Similarly, for $H \in \mathcal{H}^\circ(\mu, \nu, \lambda)$, its first derived subarray $T_1(H)$ satisfies $T_1^*(H) \in GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$ by Theorem 3.2. We can therefore identify a H such that $T_1(H) = T_Z$ and this gives us a bijection from $LR(\lambda/\mu, \nu)$ to $\mathcal{H}^\circ(\mu, \nu, \lambda)$. □

We remark that, in [Bu00], Fulton gave a bijection between LR tableaux and hives by using *contratableaux*.

Example 4.6. *To find the hive corresponding to the LR tableau given in Example 4.3, we need to determine the inner points of the* h*-array* H

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 5 & & 11 & & & \\ & & 8 & & p & & 18 & & \\ & 9 & & q & & r & & 23 & \\ 9 & & 16 & & 21 & & 24 & & 26 \end{array}$$

whose boundary values are determined by $\mu = (5, 3, 1, 0), \nu = (7, 5, 3, 2)$, *and* $\lambda = (11, 7, 5, 3)$. *It can be done by using the middle subarray* T_Z *of the corresponding truncated GT pattern*

$$\begin{array}{ccccccc} 5 & & 3 & & 1 & & 0 \\ & 4 & & 1 & & 0 & \\ & & 2 & & 1 & & \\ & & & 1 & & & \end{array}$$

as the subarray T_1, *i.e., the differences along NE-SW diagonals of* H. *This gives us* $p = 15$, $q = 16$, *and* $r = 20$.

5. LR Tableaux and GT Patterns II

In this section, we show that the semistandard and Yamanouchi conditions for tableaux are equivalent to, respectively, the exponent and semistandard conditions for their *companion tableaux*. This correspondence is obtained by taking the transpose of matrices describing tableaux. As a result, we show that the companion tableaux of LR tableaux are GZ schemes under the tableau-pattern bijection.

5.1. For a (skew) semistandard tableau Y, as in (4.1.1), we let $a_{i,j}(Y)$ denote the number of i's in the jth row.

Definition 5.1. *For a (skew) semistandard tableau* Y*, its companion tableau* Y^c *is defined as a non-skew tableau whose entries are weakly increasing along each row and whose number of* i*'s in the* j*th row is equal to* $a_{j,i}(Y)$*; that is, for* $1 \le i, j \le n$*,*

$$a_{i,j}(Y^c) = a_{j,i}(Y). \tag{5.1.1}$$

Example 5.2. *For the LR tableau* Y *from Example 4.3, the associated matrix is*

$$a_{i,j}(Y) = \begin{bmatrix} 6 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Then, from its transpose, we have the following companion tableau Y^c*.*

1	1	1	1	1	1	2
2	2	2	3	4		
3	3	3				
4	4					

Note that Y *is of shape* $(11,7,5,3)/(5,3,1,0)$ *and content* $(7,5,3,2)$ *while its companion tableau* Y^c *is of shape* $(7,5,3,2)$ *and content* $(6,4,4,3)$*, which is* $(11,7,5,3) - (5,3,1,0)$*. The GT pattern* T_{Y^c} *corresponding to* Y^c *is*

$$\begin{array}{ccccccc} 7 & & 5 & & 3 & & 2 \\ & 7 & & 4 & & 3 & \\ & & 7 & & 3 & & \\ & & & 6 & & & \end{array}.$$

We want to show that this correspondence $Y \mapsto T_{Y^c}$ *gives another bijection from the set of LR tableaux to the set of GZ schemes.*

In [vL01], van Leeuwen replaced the Yamanouchi condition in LR tableaux with the semistandard condition in their companion tableaux. Here, we show that the semistandard condition in LR tableaux has a counterpart in the companion tableaux as well, and then we identify the companion tableaux as an independent object equivalent to GZ schemes.

Theorem 5.3. *For a LR tableau* Y*, we let* Y^c *denote its companion tableau and let* T_{Y^c} *denote the GT pattern corresponding to* Y^c*. The map* $\psi(Y) = T_{Y^c}$ *gives a*

bijection from $LR(\lambda/\mu, \nu)$ *to* $GZ(\nu, \lambda - \mu, \mu)$. *In particular, the Yamanouchi and semistandard conditions in* Y *are equivalent to, respectively, the interlacing condition IC(2) and the exponent condition in* T_{Y^c}.

Proof. From (5.1.1), Y is a tableau of shape λ/μ if and only if the content of Y^c is equal to $\lambda - \mu$. The content of Y is equal to the shape of Y^c. The type and weight of T_{Y^c} are therefore ν and $\lambda - \mu$, respectively.

Recall the Yamanouchi condition in Y (4.3.3): for $0 \leq i < n$ and $1 \leq j < n$,

$$\sum_{k=1}^{i} a_{j,k}(Y) \geq \sum_{k=1}^{i+1} a_{j+1,k}(Y). \tag{5.1.2}$$

Since $a_{i,j}(Y) = a_{j,i}(Y^c)$ for all i and j, this inequality, in terms of the entries in Y^c, is saying that the number of entries less than or equal to $i + 1$ in the $(j + 1)$th row is not more than the number of entries less than or equal to i in the jth row. It is the semistandard condition for Y^c and therefore the interlacing condition for T_{Y^c}.

To show this, consider expressing the elements of the GT pattern $T_{Y^c} = (t_j^{(i)})$ in terms of $a_{i,j}(Y^c)$. From the standard bijection between semistandard tableaux and GT patterns, (4.1.3), we have

$$t_j^{(i)} = \sum_{k=1}^{i} a_{k,j}(Y^c)$$

where $a_{i,j}(Y^c)$ is the number of i entries in the jth row of Y^c.

Consider the interlacing condition IC(2): $t_j^{(i)} \geq t_{j+1}^{(i+1)}$ where $0 \leq i < n$ and $1 \leq j < n$. Writing this with the above relation gives

$$\sum_{k=1}^{i} a_{k,j}(Y^c) \geq \sum_{k=1}^{i+1} a_{k,j+1}(Y^c) \quad \Leftrightarrow \quad \sum_{k=1}^{i} a_{j,k}(Y) \geq \sum_{k=1}^{i+1} a_{j+1,k}(Y)$$

which is exactly the expression for the Yamanouchi condition (5.1.2). It can be similarly shown that, as mentioned in Remark 4.2, IC(1) is equivalent to $a_{i,j}(Y) \geq 0$.

Using (4.3.2), the semistandard condition for Y says we have, for all $1 \leq \ell \leq n$ and $1 \leq m < n$,

$$\left(\sum_{k=1}^{\ell} a_{k,m+1}(Y) - \sum_{k=1}^{\ell-1} a_{k,m}(Y) \right) \leq (\mu_m - \mu_{m+1}) \tag{5.1.3}$$

or

$$\sum_{k=1}^{\ell-1} (a_{m+1,k}(Y^c) - a_{m,k}(Y^c)) + a_{m+1,\ell}(Y^c) \leq (\mu_m - \mu_{m+1}).$$

To finish our proof, it is enough to show that the left hand side of the above inequality is the exponent $\varepsilon_\ell^{(m)}(T_{Y^c})$. This can be easily seen, by using (4.1.4), as

$$\begin{aligned}
\varepsilon_\ell^{(m)}(T_{Y^c}) &= \sum_{1\le h<\ell}\left(t_h^{(m+1)}-2t_h^{(m)}+t_h^{(m-1)}\right)+\left(t_\ell^{(m+1)}-t_\ell^{(m)}\right)\\
&= \sum_{1\le h<\ell}\left(\sum_{k=h}^{m+1}a_{k,h}(Y^c)-2\sum_{k=h}^{m}a_{k,h}(Y^c)+\sum_{k=h}^{m-1}a_{k,h}(Y^c)\right)+\left(\sum_{k=\ell}^{m+1}a_{k,\ell}(Y^c)-\sum_{k=\ell}^{m}a_{k,\ell}(Y^c)\right)\\
&= \sum_{1\le k<\ell}\left(a_{m+1,k}(Y^c)-a_{m,k}(Y^c)\right)+a_{m+1,\ell}(Y^c).
\end{aligned}$$

□

We now have an alternative proof of Corollary 4.5.

Corollary 5.4. *There is a bijection between* $\mathsf{LR}(\lambda/\mu,\nu)$ *and* $\mathcal{H}^\circ(\mu,\nu,\lambda)$.

Proof. We can map any $Y\in\mathsf{LR}(\lambda/\mu,\nu)$ to $T_{Y^c}\in GZ(\nu,\lambda-\mu,\mu)$ via the bijection in Theorem 5.3. From Theorem 3.2 there is a bijection between $\mathcal{H}^\circ(\mu,\nu,\lambda)$ and $GZ(\nu,\lambda-\mu,\mu)$ through the derived t-array T_2 of a hive. The composition of the first bijection with the inverse of the second then gives a bijection which assigns $Y\in\mathsf{LR}(\lambda/\mu,\nu)$ to $H\in\mathcal{H}^\circ(\mu,\nu,\lambda)$ if and only if $T_2(H)=T_{Y^c}$. □

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